A NOTE ON SOME PARTITIONS RELATED TO TERNARY QUADRATIC FORMS

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Abstract. We offer some partition functions related to ternary quadratic forms, and note on its asymptotic behavior. We offer these results as an application of a simple method related to conjugate Bailey pairs.

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1. Introduction

We write a positive ternary quadratic form as

$$f(x,y,z)=ax^2+by^2+cz^2+ryz+szx+txy, \tag{1.1}$$

and say it is primitive if $gcd(a,b,c,r,s,t)=1$. We recall some elementary facts about $f(x,y,z)$. We first consider $(a,b,c,r,s,t)=(1,1,1,0,0,0)$. The number $S(X)$ of integers less than or equal to X which are representable as a sum of three squares is $\sim \frac{5}{6}X$. (See [7] for an interesting discussion on this result and related material.) Here we used the common notation $F(x)\sim G(x)$ to mean that $F(x)$ is asymptotically equivalent to $G(x)$, or $\lim_{x\to\infty} F(x)/G(x)=1$. We will also use Vinogradov's $F(x)\ll G(x)$ to mean that $|F(x)|\le \sigma G(x)$ for some constant $\sigma\ge 0$. In general, it is known that

$$\#\{n\le X : n=f(x,y,z)\}\sim \sigma' X, \tag{1.2}$$

where $\sigma'\in(0,1]$. Fomenko [3] gives a nice introduction on primitive quadratic forms in $k\ge 2$ variables. In particular, we recommend the discussion mentioned there on ternary quadratic forms in relation to (1.2). See Blomer and Granville [2] for a discussion on estimates for quadratic forms and a discussion on Bernay's result that

$$\#\{n\le X : n=ax^2+by^2+cxy\}\sim \lambda\frac{X}{\sqrt{\log X}}, \tag{1.3}$$

for a positive constant λ.

In [5] we find $\ll$ results for partition functions related to (1.3), offering some interesting applications of q-series arising from Bailey pairs associated with indefinite binary quadratic forms. A simple method used in [6], which uses a definition of a Bailey pair in conjunction with conjugate Bailey pairs, produced the tools needed to obtain q-series related to ternary quadratic forms. The motivation of this paper is to obtain partition functions which are related to ternary forms with gcd equal to 1, similar to $S(X)$. In this way, we offer $\ll$ results and offer further analysis on the main partition function considered herein.

2. Bailey's Lemma and identities

Here we discuss the analytic tools used to obtain our generating functions. Put [4] $(Z;q)_n = (Z)_n = (1-Z)(1-Zq)\cdots(1-Zq^{n-1})$. Bailey [1] introduced the idea of a pair of sequences (α_n, β_n), relative to a, which satisfy

$$\beta_n(a,q) = \beta_n = \sum_{r\geq 0} \frac{\alpha_r(a,q)}{(aq)_{n+r}(q)_{n-r}}. \tag{2.1}$$

Further formulas needed for our study are a specialization of Bailey's lemma [1] (relative to $a=q$)

$$\sum_{n\geq 0}(q)_n(-1)^n\beta_n q^{n(n+1)/2} = (1-q)\sum_{n\geq 0}(-1)^n q^{n(n+1)/2}\alpha_n, \tag{2.2}$$

and the pairs [6, eq(3.5)] $(\alpha_n(q^2,q^2), \beta_n(q^2,q^2))$,

$$\beta_n(q^2,q^2) = \frac{q^{-n}}{(-q)_{2n}}, \tag{2.3}$$

$$\alpha_n(q^2,q^2) = (-1)^n q^{n(n-1)}\frac{(1-q^{4n+2})}{1-q^2}\sum_{i\geq 0} q^{i(i+1)/2}\sum_{2|j|\leq i}(-1)^j q^{-j(j-1)+2nj}, \tag{2.4}$$

and [6, eq(3.6)] (α_n, β_n),

$$\beta_n = \frac{(q)_n(-1)^n q^{n(n-1)/2}}{(q)_{2n}}, \tag{2.5}$$

$$\alpha_n = (-1)^n q^{n(n-1)/2}\frac{(1-q^{2n+1})}{1-q}\sum_{i\geq 0}(-1)^i q^{i(3i+1)/2}(1-q^{2i+1})\sum_{|j|\leq i}(-1)^j q^{-j(j-1)/2+nj}. \tag{2.6}$$

Inserting (2.3)–(2.4) and (2.5)–(2.6) into (2.2) gives us the following lemma containing our needed identities, which will be used in the next section.

Lemma 2.1. *We have,*

(2.7)
$$\sum_{n\geq 0}\frac{(q^2;q^2)_n(-1)^nq^{n^2}}{(-q)_{2n}}=\sum_{n\geq 0}q^{2n^2}(1-q^{4n+2})\sum_{i\geq 0}q^{i(i+1)/2}\sum_{2|j|\leq i}(-1)^jq^{-j(j-1)+2nj},$$

and

(2.8)
$$\sum_{n\geq 0}\frac{(q)_nq^{n^2}}{(q^{n+1})_n}=\sum_{n\geq 0}q^{n^2}(1-q^{2n+1})\sum_{i\geq 0}(-1)^iq^{i(3i+1)/2}(1-q^{2i+1})\sum_{|j|\leq i}(-1)^jq^{-j(j-1)/2+nj}.$$

3. Partitions

We are concerned with one main generating function from which our partition theorems will follow.

Lemma 3.1. *Let $A_{k,m}(n)$ be the number of partitions of n where: (i) k appears at most twice. (ii) All parts $< k$ appear at least twice and at most thrice. (iii) m is the number of parts $> k$ and $\leq 2k$. (iv) All parts are $\leq 2k$, and parts that are $\geq k+1$ and $\leq 2k$ may appear any number of times. Further, let $\bar{A}_{k,m}(n)$ be those partitions counted by $A_{k,m}(n)$ with number of parts that are $\leq k$ odd minus those with number of parts that are $\leq k$ even. Then,*

(3.1)
$$\sum_{n,m\geq 0}\bar{A}_{k,m}(n)a^mq^n=\frac{(q)_kq^{1+1+2+2+\cdots+(k-1)+(k-1)+k}}{(1-aq^{k+1})(1-aq^{k+2})\cdots(1-aq^{2k})}=\frac{(q)_kq^{k(k+1)/2+k(k-1)/2}}{(aq^{k+1})_k}.$$

We write out the numerator of the right side of (3.1) for the reader to show the weight associated with parts that are $\leq k$ for the sake of clarity. Define the polynomial in x by $f_k(x):=(x;x)_kx^{1+1+2+2+\cdots+(k-1)+(k-1)+k}$. Then

$$f_1(x)=x^1-x^{1+1}. \tag{3.2}$$

$$f_2(x)=x^{1+1+2}-x^{1+1+1+2}-x^{1+1+2+2}+x^{1+1+1+2+2}. \tag{3.3}$$

(3.4)
$$f_3(x)=x^{2+2+1+1+3}+x^{1+1+1+2+2+2+3}+x^{1+1+1+2+2+3+3}$$
$$+x^{1+1+2+2+2+3+3}-x^{1+1+1+2+2+3}-x^{1+1+2+2+2+3}-x^{1+1+2+2+3+3}-x^{1+1+1+2+2+2+3+3}.$$

It is seen from (3.2)–(3.4) that the weight is $+1$ when the number of parts is odd and -1 if the number of parts is even.

Put

$$\sum_{m,k\geq 0}\bar{A}_{k,m}(n)=B(n), \tag{3.5}$$

and

$$\sum_{m,k\geq 0}(-1)^{m+k}\bar{A}_{k,m}(n)=\bar{B}(n). \tag{3.6}$$

Theorem 3.2. *We have,*

$$\sum_{n\geq 0}\frac{(q^2;q^2)_n(-1)^nq^{n^2}}{(-q)_{2n}}=\sum_{n\geq 0}\frac{(q)_n(-1)^nq^{n(n+1)/2+n(n-1)/2}}{(-q^{n+1})_n}=\sum_{n\geq 0}\bar{B}(n)q^n, \tag{3.7}$$

$$\sum_{n\geq 0}\frac{(q)_nq^{n^2}}{(q^{n+1})_n}=\sum_{n\geq 0}B(n)q^n. \tag{3.8}$$

As a natural consequence of our partition function having an intimate connection with a primitive ternary quadratic form, we are able to state the following two properties.

Corollary 3.2.1. *We have that,*

$$\#\{n\leq X: B(n)\neq 0\}\ll X,$$

and

$$\#\{n\leq X: \bar{B}(n)\neq 0\}\ll X.$$

Further we have that both $B(n)$ and $\bar{B}(n)$, admit an expansion of the form $\sum_{m\in S_1}\omega_1(m,n)+\sum_{m\in S_2}\omega_2(m,n)$, for some weight functions $\omega_i(m,n)$, $i=1,2$. Here S_1 is the set of solutions to $m=x^2+xy+y^2$, and S_2 is the set of triples which satisfy the summation bounds obtained from taking q^n in (2.8), respectively (2.7), but with strict inequality (i.e. $<$).

Proof. The first two claims follow from elementary facts about primitive quadratic forms in 3 variables. The expansion claim follows from taking the ternary q-series on the right side of (2.8), and splitting the sum of triples (n,i,j) into two q-series, one with (n,i,i) and the other with $i\neq j$, (n,i,j). The one with $i=j$ reduces to a

q-series of the form $\sum q^{x^2+xy+y^2}$. In the case of (2.7), the process is the same but splitting the sum when $i = 2j$. $\square$

We note that Bernay's theorem says that the first part of the expansion $\sum_{m\in S_1} \omega_1(m,n)$ has arithmetic density 0.

Since attaching a weight with potential cancellation of terms in the ternary form expansion makes the set smaller, we may apply the $\ll$. If it were the case that the ternary expansions had no weight (i.e. non-negative coefficients), we would have that these sets are also $\gg X$. In the instance where this is true, one may apply the squeeze theorem to obtain a $\sim cX$ result similar to (1.2). But we know that our ternary expansions have some amount of cancelation of terms, and so there is some $\phi_1(X) \leq X$ such that $\phi_1(X) \ll \#\{n \leq X : B(n) \neq 0\}$, for example. Therefore, this opens the possibility that $B(n)$ (and $\bar{B}(n)$) could turn out to be lacunary, since it may also be the case that there exists a function $\phi_2(X) \leq X$, where $\#\{n \leq X : B(n) \neq 0\} \ll \phi_2(X)$.

4. Further Comments

We state some motivation for further research concerning $B(n)$ and $\bar{B}(n)$ as well as other partitions associated with ternary forms like the ones herein. First we note that the second sum in (2.7) may be written in the form

$$\sum_{n\geq 0} q^{2n^2}(1-q^{4n+2}) \sum_{i\geq 0, j\in\mathbb{Z}} (-1)^j q^{i(i+1)/2+j(j+1)+(2i+1)|j|+2jn}, \tag{4.1}$$

Secondly, the second sum in (2.8) may similarly be rewritten as a q-series related to a positive ternary quadratic form with $gcd = 1$. Identities (2.7) (respectively (2.8)) tell us that a necessary condition for $\bar{B}(n)$ (respectively $B(n)$) to be nonzero is when n is represented by a ternary quadratic form. In light of (1.2), it remains to show whether the following two statements are true (or false).

Conjecture 4.1. $\#\{n \leq X : B(n) \neq 0\} \sim \sigma_1 X$, *where* $\sigma_1 \in (0,1]$.

Conjecture 4.2. $\#\{n \leq X : \bar{B}(n) \neq 0\} \sim \sigma_2 X$, *where* $\sigma_2 \in (0,1]$.

In general, it is desirable to be able to classify the arithmetic density of partitions according to the weight associated with the quadratic expansion. While Theorems 3.3 and 3.4 provide us with a stepping stone, it is an open problem to find the

upper arithmetic density $\lim_{n\to\infty}\sup \#\{n \le X : B(n) \neq 0\}/n$.

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